Analog-to-digital converters (ADCs) and sampling theory

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

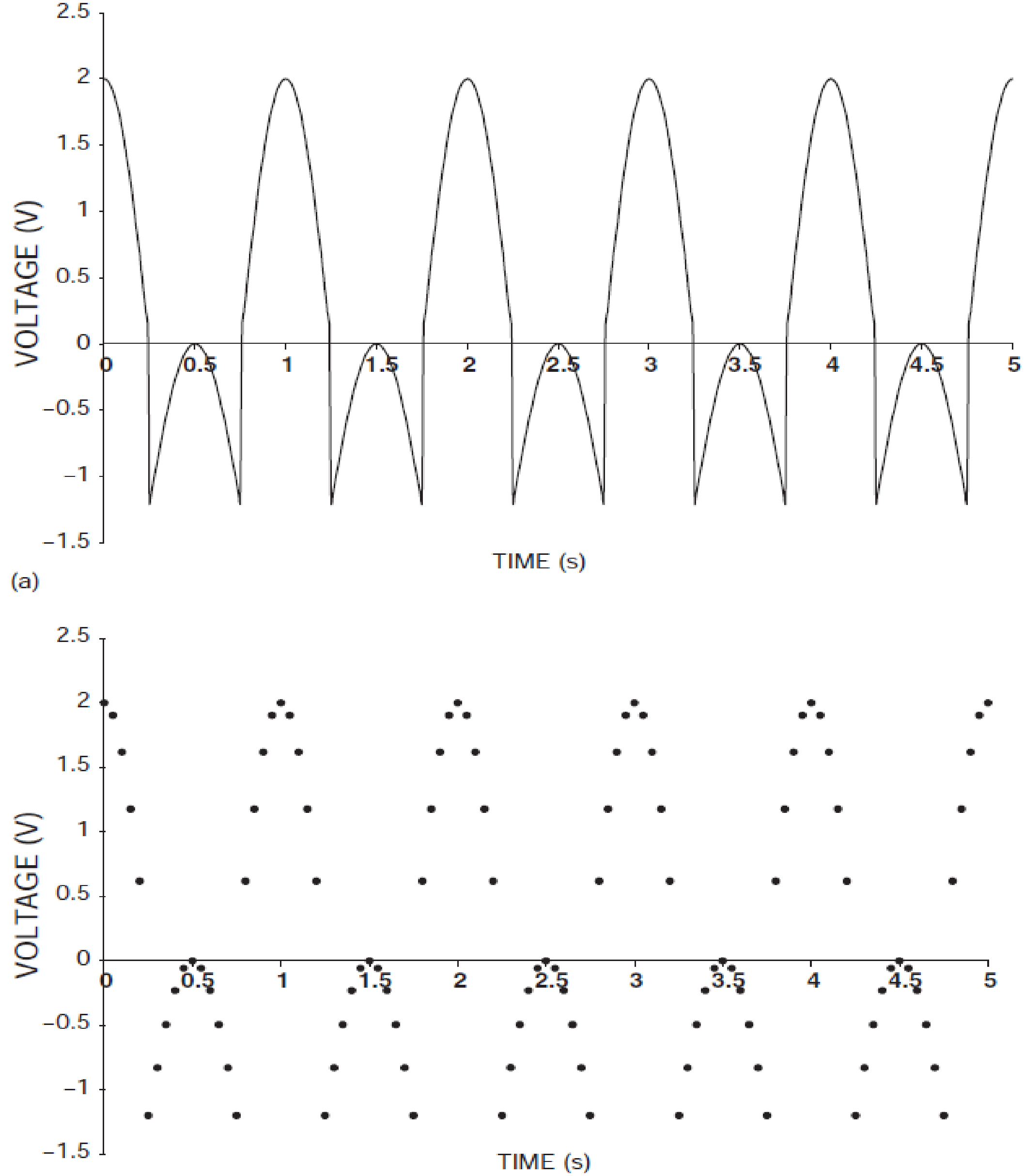
Analog-to-digital conversion

- Digital computers are intrinsically discrete-time, finite-resolution devices.
- Signals intended for analysis must therefore be converted to a discrete series of numbers.
- Analog-to-digital (A/D) converters are used to transform biological signals from continuous analog waveforms to digital sequences.

Analog-to-digital conversion

- An A/D converter is a computer-controlled voltmeter, which measures an input analog signal and gives a numeric representation of the signal as its output.
- Analog-to-digital conversion is called digitization, which has two parts
  - **Sampling:** determines the time points at which data are measured.
  - **Quantization:** determines how the continuous amplitude is converted to a number (independent of time).

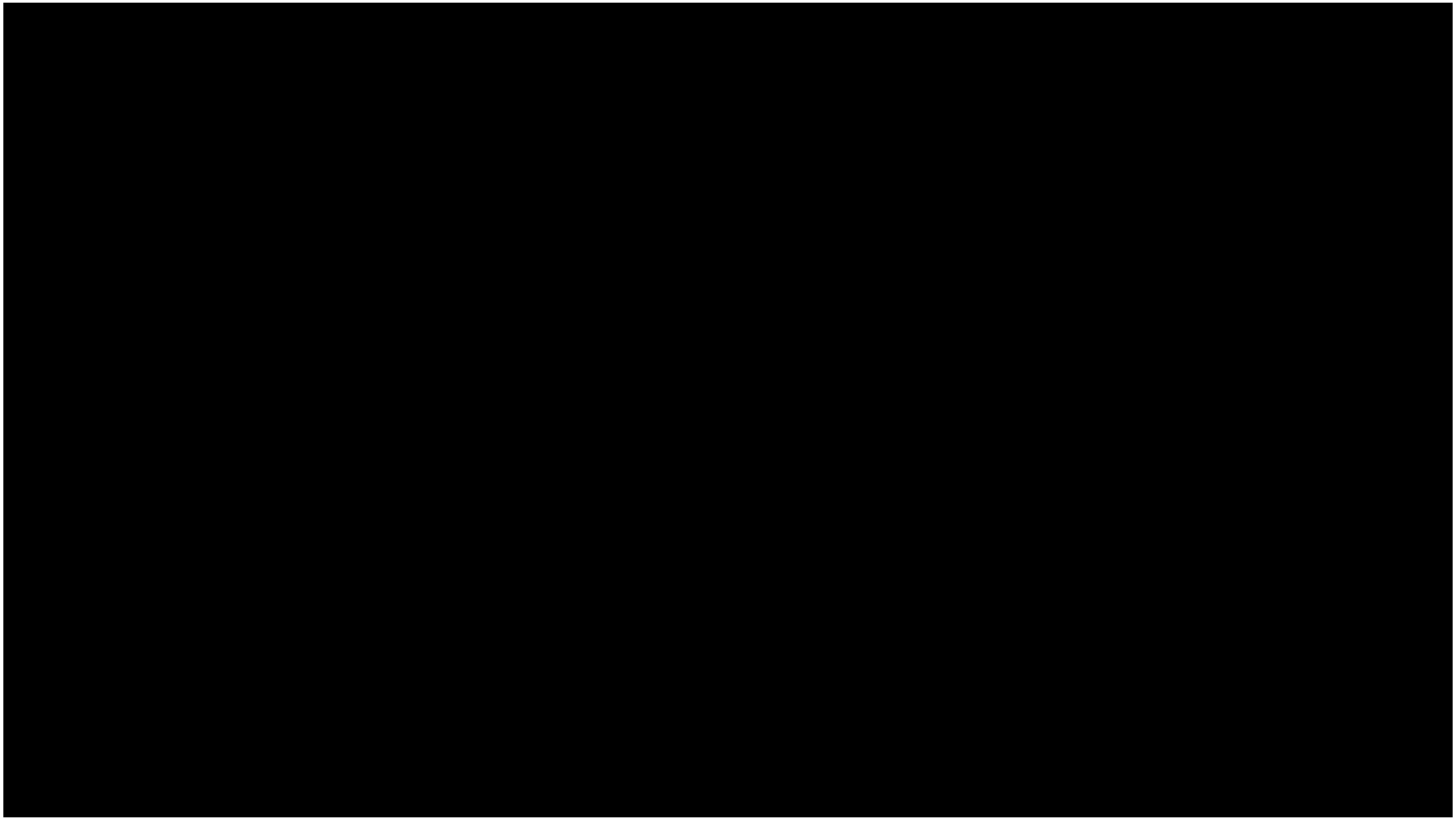
Example of digitization



Sampling theory

- Many physiological processes generate signals that are continuous functions of time and have continuous amplitude, or continuous distribution of amplitude.
- Given a signal, *x(t)*, sampling theory asks whether it is possible to collect information on *x(t)* at discrete points $t0$, $t1$, $t2$, and so on, and still have captured all the information contained in *x(t)*.

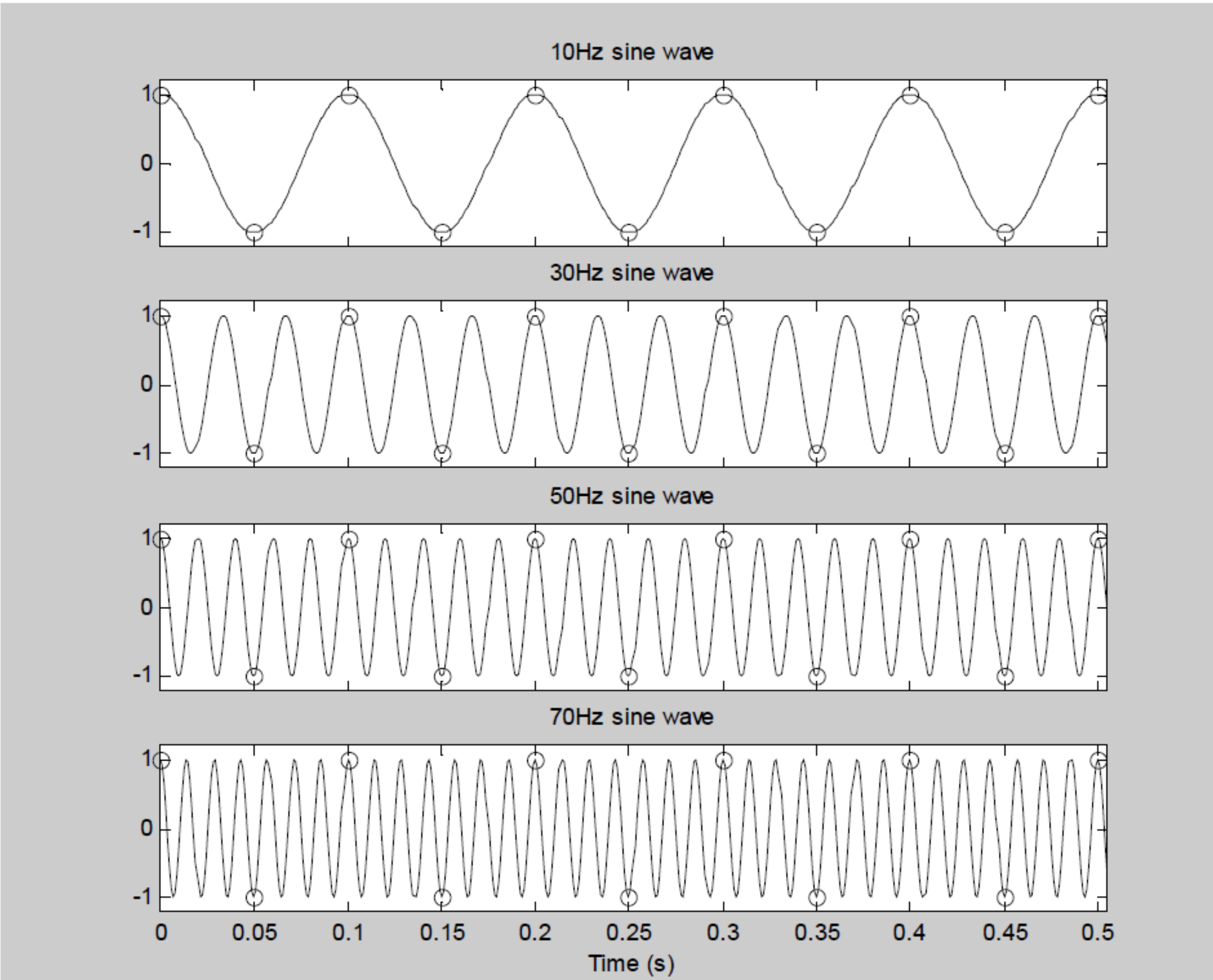
Wagon-wheel effect



https://www.youtube.com/watch?v=VNftf5qLpiA

Aliasing

- Aliasing is the confusion of high- and low-frequency components in the original analog signal



F_s=20 Hz

Aliasing

- For a signal sampled at a rate of f_s samples/second, the frequencies given by

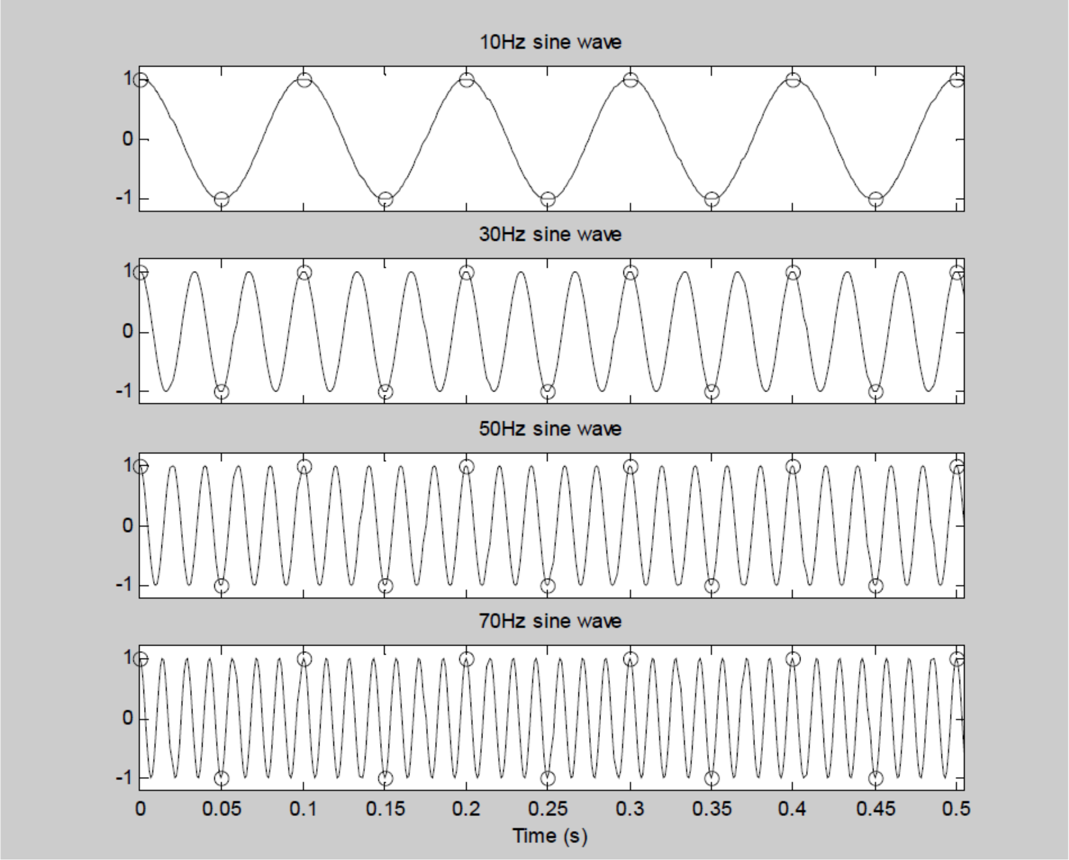
$$f_i = nf_s \pm f \text{ for } n = 1, 2, 3, \ldots$$

$$\text{where } 0 \leq f \leq \frac{f_s}{2}$$

cannot be distinguished from each other.

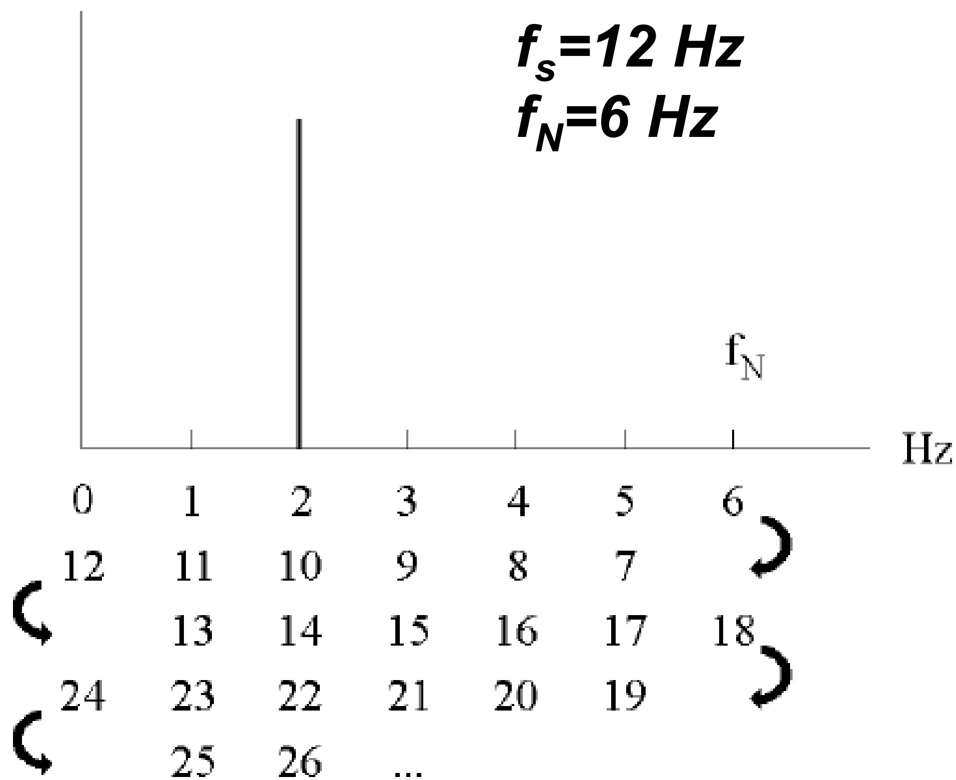
Aliasing

- With a sampling rate of 20 Hz, components with frequencies of 10, 30, 50, 70, etc. will all yield exactly the same sample values, since they are aliased to 10 Hz.

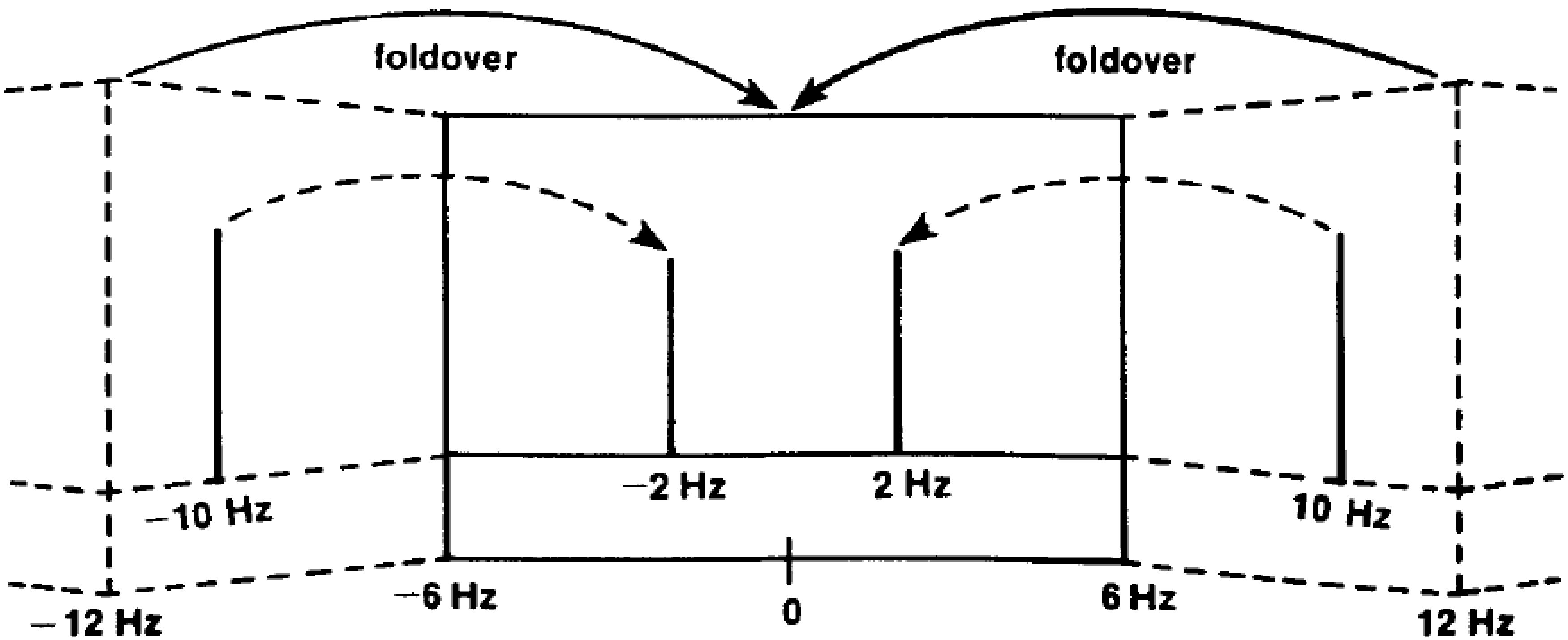


Aliasing

- The half of the sampling frequency f_N (i.e. Nyquist frequency) is sometimes called the folding frequency because frequencies above the Nyquist frequency are 'folded back' to below f_N.
- Aliased frequency will be out-of-phase to original frequency



Frequency folding



Nyquist sampling theorem

- To unambiguously capture all information contained in the band-limited time domain signal *x(t)* having bandwidth *W,* it is necessary to sample at a rate f_s of at least 2*W*.
- In other words, we have to sample at least twice the frequency of the highest spectral component in *x(t)*, i.e. mathematically stated: $fs > 2W$.

Delta Function

- Delta function is a non-physical, singularity function

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \text{undefined} & \text{at } x = 0 \end{cases}$$

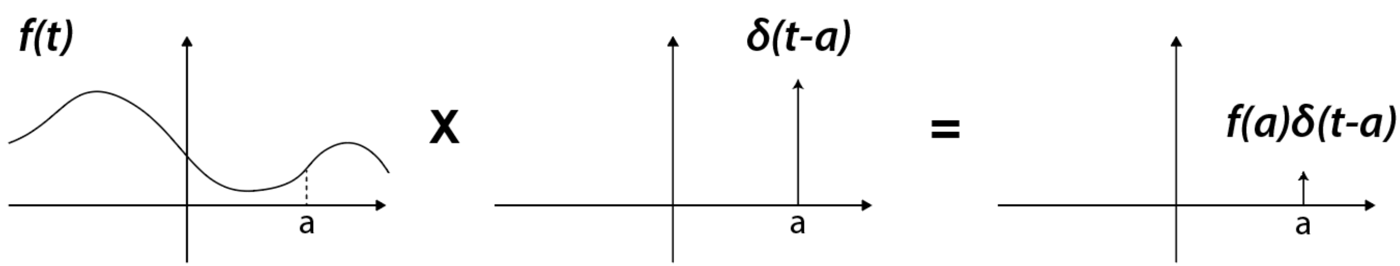
$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

- Sampling property

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

$$f(t) \times \delta(t-a) = f(a) \times \delta(t-a)$$

$$= \begin{cases} f(a) & when\ t = a \\ 0 & when\ t \neq a \end{cases}$$



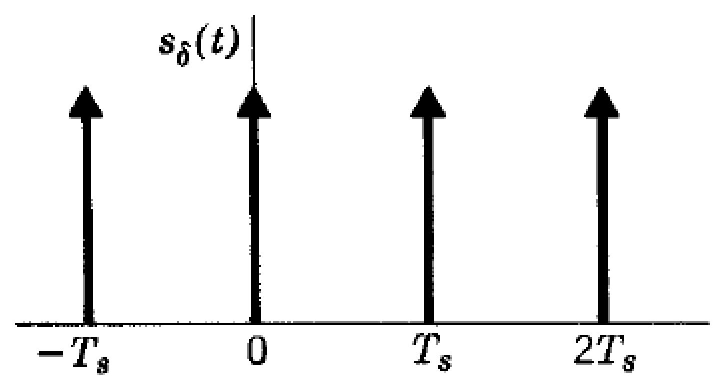
Ideal sampling function

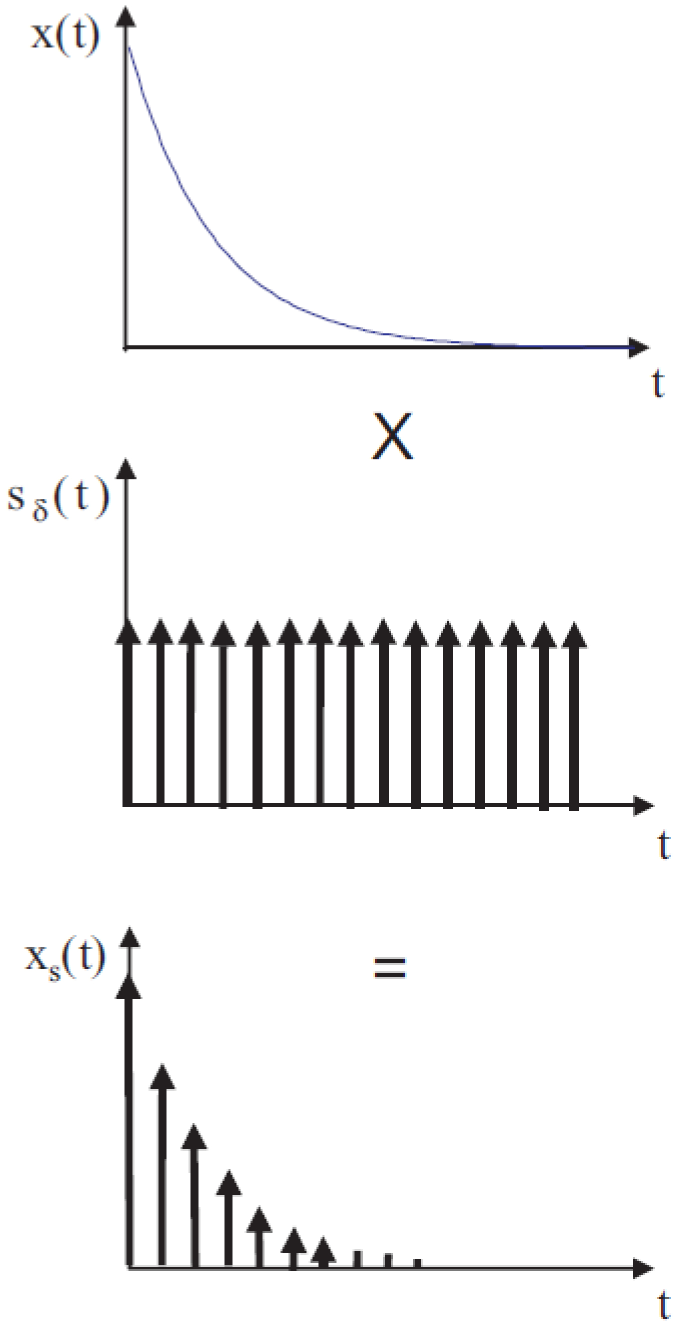
- The mathematical representation of the sampled version $x_s(t)$ of a continuous time signal $x(t)$ is given by

$$x_s(t) = x(t) \times s_\delta(t)$$

where $s_\delta(t)$ is known as the ideal sampling function and is defined as

$$s_\delta(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$





What is the representation of $x_s(t)$ in frequency domain, i.e. the frequency spectrum of $x_s(t)$?

$$\Im\{x(t)\cdot y(t)\} = \Im\{x(t)\} * \Im\{y(t)\}$$

$$\downarrow$$

$$\begin{aligned} X_s(f) &\equiv \Im\{x_s(t)\} \\ &= \Im\{x(t)\} * \Im\{s_\delta(t)\} \\ &\equiv X(f) * S_\delta(f) \end{aligned}$$

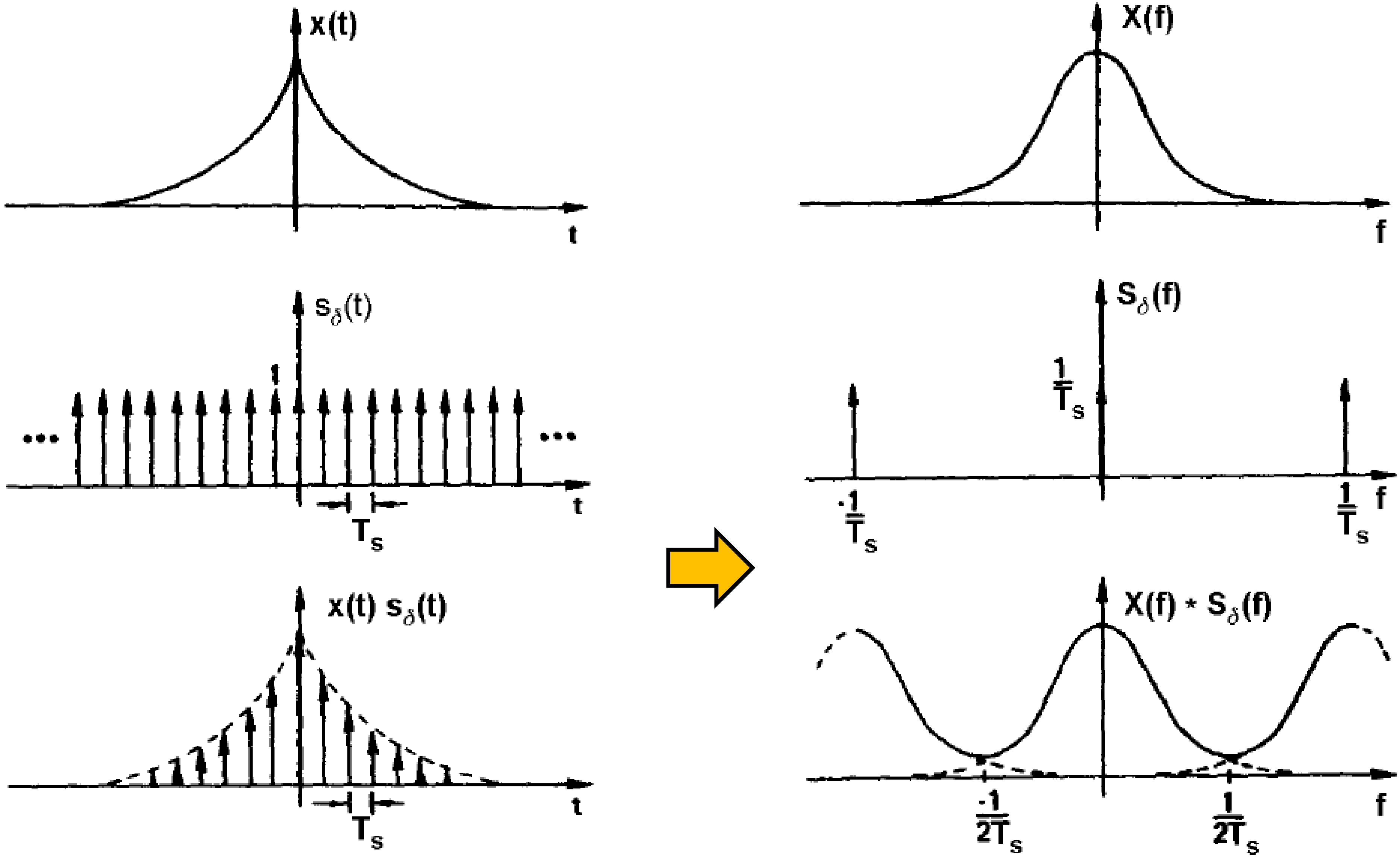
The frequency spectrum of the sampled signal

Since

$$S_\delta(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\delta(f - kf_s)$$

$$\begin{aligned}
X_s(f) &= X(f) * S_\delta(f) \\
&= X(f) * \left\{\frac{1}{T_s}\sum_{k=-\infty}^{\infty}\delta(f - kf_s)\right\} \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(f) * \delta(f - kf_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(f - kf_s)
\end{aligned}$$

The frequency spectrum of the sampled signal $x_s(t)$ is the Fourier transform of x(t), i.e. X(f) at k = 0, plus replicas of X(f) centered at ±kfs for k = 1, 2, ….


x(t)
t
s_δ(t)
1
T_s
t
x(t) s_δ(t)
T_s
t
X(f)
f
S_δ(f)
1/T_s
-1/T_s
1/T_s
f
X(f) * S_δ(f)
-1/2T_s
1/2T_s
f

Anti-aliasing filters

- Aliasing cannot be avoided during sampling.
- But one can only strive to minimize aliasing by attenuating frequencies above the Nyquist frequency prior to sampling.
- This is done by restricting the frequency range of the original data with an analog low-pass filter. Such filters are often referred to as ***anti-aliasing filters***.

Quantization

- ADCs perform quantization of an analog waveform, which is a non-linear operation that involves assigning a finite number of amplitude levels across the waveform.
- Given q, a quantizer (i.e. ADC) finds i to satisfy

$$iq - \frac{q}{2} \le x < iq + \frac{q}{2} \text{ for } i = (\ldots -2, -1, 0, 1, 2, \ldots)$$

- Quantization reduces the resolution of the original continuous waveform. This is an acceptable price to pay to be able to get data into a digital processor

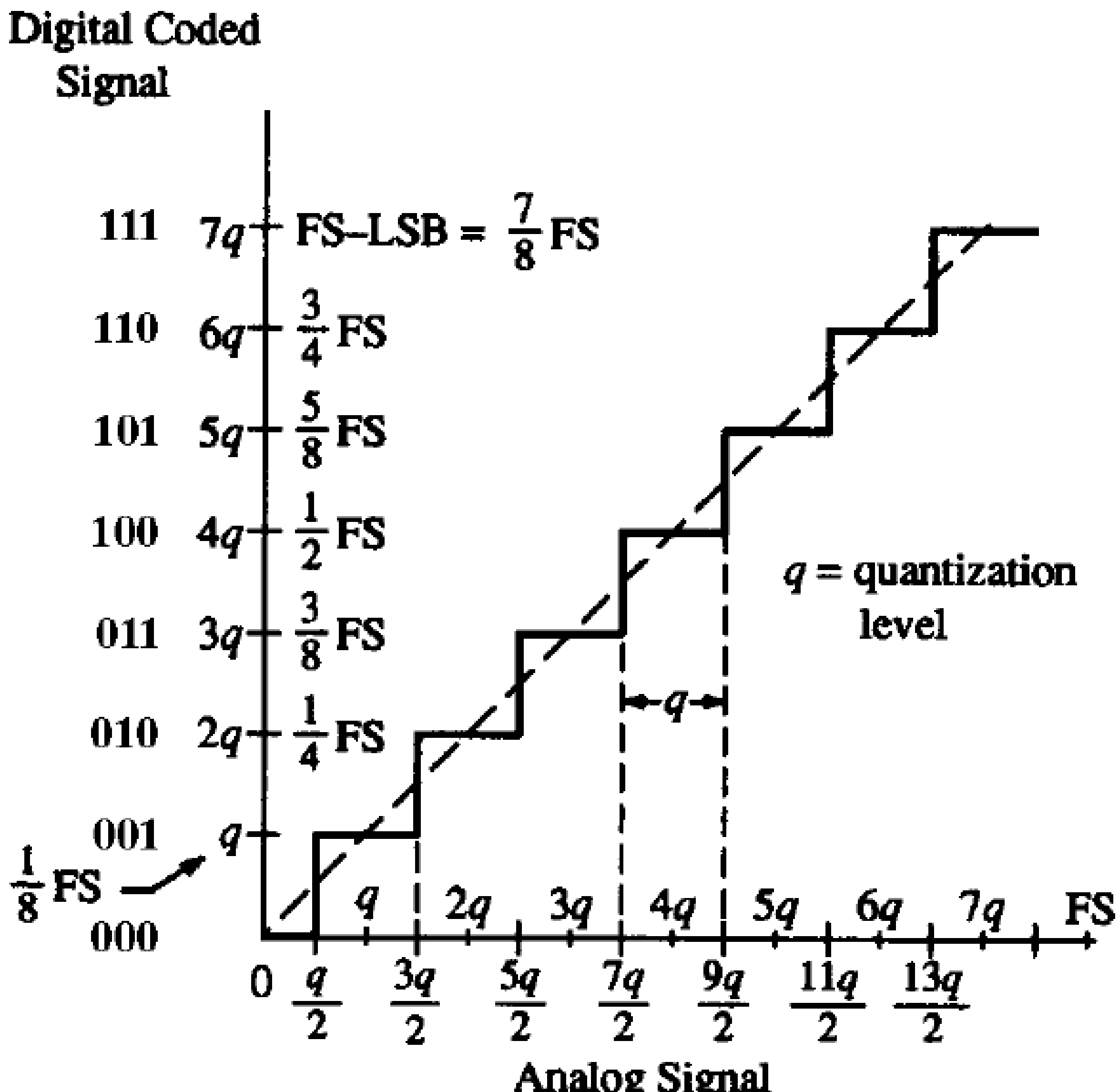
Dynamic range

- The dynamic range (DR) of an ADC refers to the range from the largest to the lowest values that can be converted without saturating the device
- Typical DR: 0-5V, 0-10V, ±5V and ±10V

Resolution

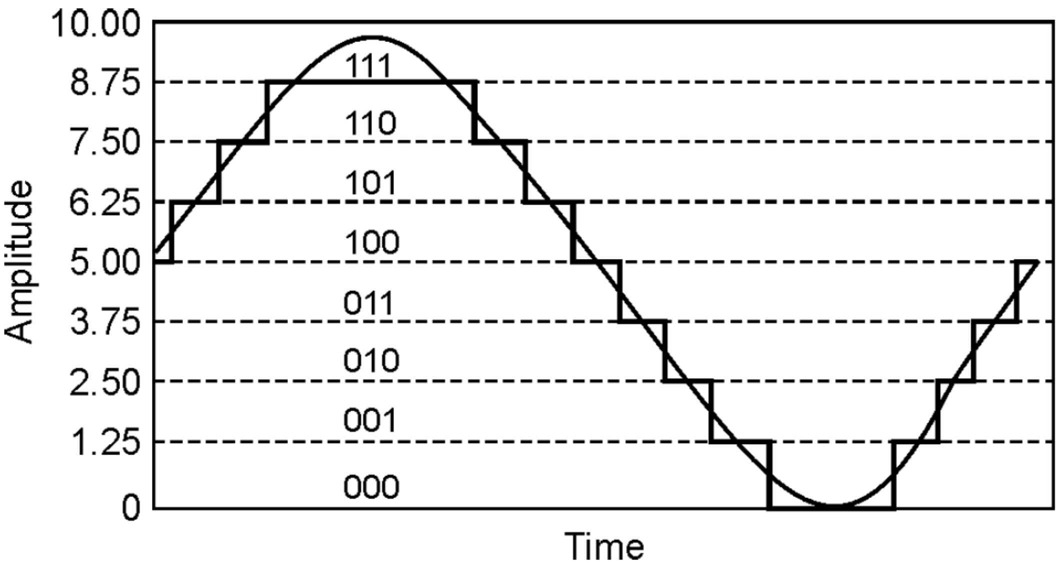
- The resolution of an ADC is most frequently expressed in the number of bits or the weight of the LSB.
- An 8 bit ADC with a DR of 0-10V will have a resolution of $10v/2^8 = 39mV$
- An 16 bit ADC with a DR of 0-10V will have a resolution of $10v/2^{16} = 0.15mV$
- A 24 bit ADC with a DR of 0-10V will have a resolution of $10v/2^{24} = 0.6uV$

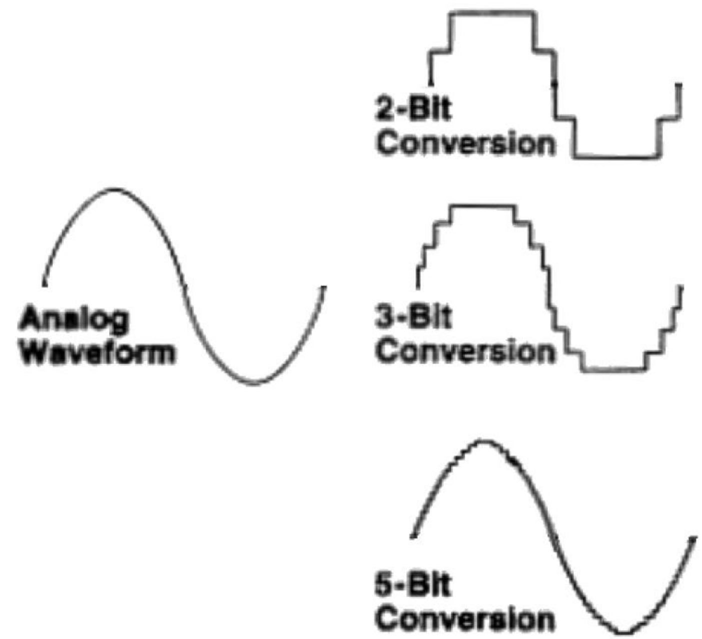
Resolution



Digital Coded Signal
111
110
101
100
011
010
001
000
7q
6q
5q
4q
3q
2q
q
FS–LSB = 7/8 FS
3/4 FS
5/8 FS
1/2 FS
3/8 FS
1/4 FS
1/8 FS
q = quantization level
q
q
2q
3q
4q
5q
6q
7q
FS
0
q/2
3q/2
5q/2
7q/2
9q/2
11q/2
13q/2
Analog Signal

Resolution

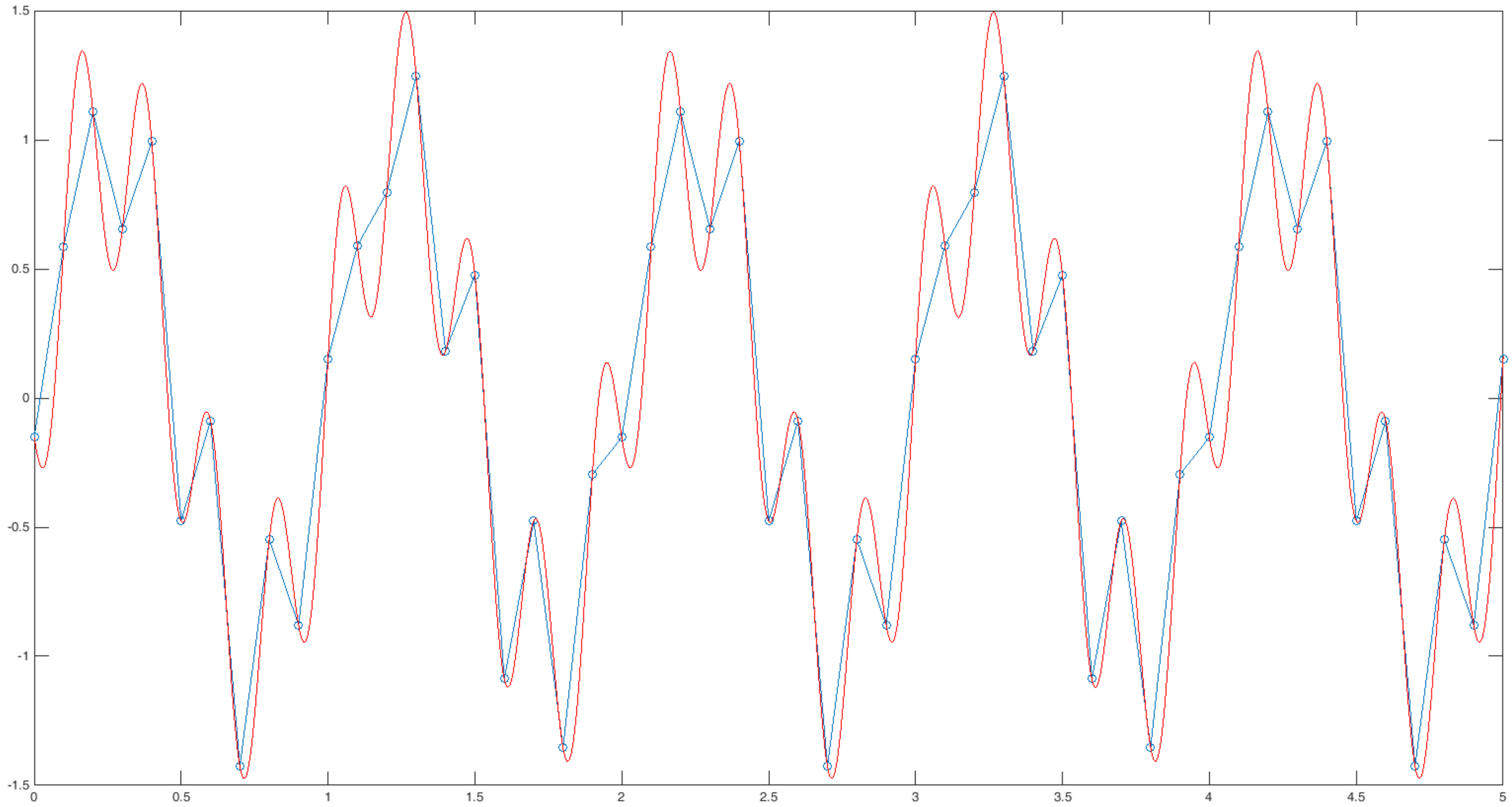

10.00
8.75
7.50
6.25
5.00
3.75
2.50
1.25
0
Amplitude
111
110
101
100
011
010
001
000
Time


Analog Waveform
2-Bit Conversion
3-Bit Conversion
5-Bit Conversion

How to reconstruct data?

Red: original data

Blue: Sampled data



How to reconstruct data?

Theorem: If the Fourier transform $F(\omega)$ of a signal function $f(t)$ is zero for all frequencies above $|\omega| \geq \omega_c$, then $f(t)$ can be uniquely determined from its sampled values

$f_n = f(nT)$

These values are a sequence of equidistant sample points spaced $\dfrac{1}{2fc} = \dfrac{Tc}{2} = T$ apart. $f(t)$ is thus given by

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$

Reconstruction of original data

Red: original data

Blue: Sampled data

Green: reconstructed data